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On Anosov Energy Levels of Hamiltonians on Twisted Cotangent Bundles

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Abstract. Let T^*M denote the cotangent bundle of a manifold M endowed with a twisted symplectic structure [**1**]. We consider the Hamiltonian flow generated (with respect to that symplectic structure) by a convex Hamiltonian $H: T^*M \to \mathbb{R}$, and we consider a compact regular energy level of H, on which this flow admits a continuous invariant Lagrangian subbundle E. When $\dim M \geq 3$, it is known [**9**] that such energy level projects onto the whole manifold M, and that E is transversal to the vertical subbundle. Here we study the case $\dim M = 2$, proving that the projection property still holds, while the transversality property may fail. However, we prove that in the case when E is the stable or unstable subbundle of an Anosov flow, both properties hold.

1. Results

Let M^n be a connected manifold without boundary, T^*M its cotangent bundle, $\pi: T^*M \to M$ the canonical projection and, if $\theta \in T^*M$, let $V(\theta) \subset T_\theta T^*M$ be the vertical fibre at θ, defined as usual as the kernel of

$$d\pi_\theta: T_\theta T^*M \to T_{\pi(\theta)}M.$$

Denote by ω_0 the canonical symplectic form of T^*M. If Ω is a closed 2-form on M then, $\omega = \omega_0 + \pi^*\Omega$ defines a new symplectic form on T^*M and the symplectic manifold (T^*M, ω) is called a *twisted cotangent bundle* [**1**]. The vertical subspaces $V(\theta)$ are Lagrangian subspaces with respect to ω for every $\theta \in T^*M$.

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Let H be a Hamiltonian on T^*M, let $J\nabla H$ be its symplectic gradient respect to ω and let ϕ_t denote its associated Hamiltonian flow; ϕ_t leaves all the level sets $\Sigma_\sigma \stackrel{\text{def}}{=} H^{-1}(\sigma)$ invariant.

Recall that a Hamiltonian $H: T^*M \to \mathbb{R}$ is said to be *convex* if for each $q \in M$ the function $H(q, \cdot)$ regarded as a function on the linear space T_q^*M has positive definite Hessian.

Let Σ_σ be a compact regular energy level. We say that $\phi_t|_{\Sigma_\sigma}$ admits a *continuous invariant Lagrangian subbundle* if there exists a continuous subbundle E of $T(T^*M)|_{\Sigma_\sigma}$ such that for all $\theta \in \Sigma_\sigma$, the fibre $E(\theta)$ is a Lagrangian subspace of $T_\theta T^*M$ and

$$E(\phi_t(\theta)) = d\phi_t(E(\theta))$$

for all $t \in \mathbb{R}$. It is well known that if the flow $\phi_t|_{\Sigma_\sigma}$ is Anosov then, both the stable and unstable subbundles are continuous invariant Lagrangian subbundles.

Motivated by the results in [**6, 8**], we proved in [**9**] the following theorem:

Theorem 1.1. *Let Σ_σ be a compact regular energy level and suppose that $\phi_t|_{\Sigma_\sigma}$ admits a continuous invariant Lagrangian subbundle E. If H is convex and ω^{n-1} is exact then,*
(a) $E(\theta) \cap V(\theta) = \{\theta\}, \quad \forall\theta \in \Sigma_\sigma$.
(b) $\pi(\Sigma_\sigma) = M$. *In particular M is compact.*

For $n \geq 3$, the form ω^{n-1} is always exact since $H^{2n-2}(T^*M, \mathbb{R}) = 0$; however if $n = 2$ this is no longer the case unless Ω is exact (note that ω_0 is always exact). Thus a natural question arises: is Theorem 1.1 still true if $n = 2$ and Ω is non-exact?

The theorems below describe the situation for $n = 2$. In what follows, we will assume without loss of generality that M^2 is compact and orientable, otherwise any closed 2-form on M is exact [**5**, Proposition IX, Section 5.13].

Theorem A. *Let $n = 2$ and let Σ_σ be a compact regular energy level and suppose that $\phi_t|_{\Sigma_\sigma}$ admits a continuous invariant Lagrangian subbundle E. If H is convex then, $\pi(\Sigma_\sigma) = M$.*

We now exhibit an example (compare with [7]) for which part (a) in Theorem 1.1 is false for $n = 2$ and Ω non-exact.

Example 1.2. Let $M = T^2$ and let $H: T^*T^2 \to \mathbb{R}$ be

$$H(q,p) = \frac{1}{2}(p_1^2 + p_2^2)$$

where

$$(q,p) = (q_1, q_2, p_1, p_2) \in T^*T^2 = T^2 \times \mathbb{R}^2.$$

Let $\Omega = dq_1 \wedge dq_2$.

If we identify $T_{(q,p)}T^*T^2$ with $\mathbb{R}^4$, it is easy to check that

$$J\nabla H(q_1, q_2, p_1, p_2) = (p_1, p_2, -p_2, p_1).$$

Integrating $J\nabla H$, it follows that all the orbits of ϕ_t are closed with period 2π.

Consider the vector fields on T^*T^2 defined by

$$X_{a,b}(q_1, q_2, p_1, p_2) = (a, b, 0, 0), \quad ab \neq 0.$$

One easily checks that

$$E_{a,b}(q_1, q_2, p_1, p_2) = \mathbb{R}J\nabla H(q_1, q_2, p_1, p_2) \oplus \mathbb{R}X_{a,b}(q_1, q_2, p_1, p_2)$$

is a C^∞-Lagrangian subbundle invariant under ϕ_t on any positive energy level.

Next note that for example $E_{1,0}$ intersects the vertical at any point of the form $(q_1, q_2, \pm 1, 0)$ on the level set $H(q,p) = 1/2$. This set is the union of two copies of T^2 and the Maslov index of every orbit of ϕ_t is 2. This clearly shows that part (a) in Theorem 1.1 is false for $n = 2$ and Ω non-exact.

However, if we assume the stronger hypothesis that $\phi_t|_{\Sigma_\sigma}$ is Anosov we show:

Theorem B. *Let $n = 2$ and let Σ_σ be a compact regular energy level and suppose that $\phi_t|_{\Sigma_\sigma}$ is Anosov. Then if H is convex,*

$$E(\theta) \cap V(\theta) = \{0\}, \quad \forall \theta \in \Sigma_\sigma,$$

where E denotes the stable or the unstable subbundle of ϕ_t.

2. Proofs

Proof of Theorem A. Suppose that $\pi(\Sigma_\sigma) \neq M$. Then we showed in [**9**, Proof of Theorem 1.2] that there exists some $\theta \in \Sigma_\sigma$ for which $E(\theta) \cap V(\theta) \neq \{0\}$ and therefore, that there exists a closed connected codimension one stratified submanifold $S \subset \Sigma_\sigma$, that is transversal to the flow ϕ_t. Moreover, the low dimensional stratas of S have codimension $\geq$ 3 and therefore S represents a cycle in homology, transversally oriented by the flow ϕ_t. Since ω restricted to the codimension one strata of S is a volume form and ω_0 is exact we have:

$$0 \neq \int_S \omega = \int_S \omega_0 + \int_S \pi^*\Omega = \int_S \pi^*\Omega.$$

Next note that $\int_S \pi^*\Omega$ only depends on the cohomology class of Ω. Since $\pi(\Sigma_\sigma) \neq M$, there exists an open set U in M, not intersecting $\pi(\Sigma_\sigma)$. Let $\widetilde{\Omega}$, be a 2-form cohomologous to Ω and with support contained in U. Then obviously $(\pi|_{\Sigma_\sigma})^*\widetilde{\Omega} = 0$ and thus

$$0 \neq \int_S \pi^*\Omega = \int_S \pi^*\widetilde{\Omega} = 0.$$

This contradiction completes the proof of Theorem A. $\square$

Proof of Theorem B. From Theorem A we know that Σ_σ is 3-manifold foliated by circles and by a result of E. Ghys [**4**], ϕ_t is topologically conjugate to the geodesic flow of a metric on M of constant negative curvature. It follows then, that the closure of the set of primitive closed orbits of ϕ_t in $H_1(\Sigma_\sigma, \mathbb{R})$ is the closure of a convex open set containing the origin in its interior, since the same property holds for the geodesic flow of a compact negatively curved manifold. Thus if

$$\alpha : H_1(\Sigma_\sigma, \mathbb{R}) \to \mathbb{R}$$

is any non-trivial cohomology class, there exists a closed orbit γ of ϕ_t so that $\alpha(\gamma) < 0$.

Suppose now that for some $\theta \in \Sigma_\sigma, E(\theta) \cap V(\theta) \neq \{0\}$, where E stands for the stable or the unstable subbundle of ϕ_t. Then (cf. [**9**, Proposition 3.3]) the Maslov class $\mu \in H^1(\Sigma_\sigma, \mathbb{R})$ associated with E is non-trivial. On the other the convexity of H implies that if γ is any closed orbit of ϕ_t, then $\mu(\gamma) \geq 0$ [**2, 3**]. This contradiction completes the proof of Theorem B. □

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